

# Solution Set 9

November 18, 2002

## 1 Notation

Throughout the solutions I will use the natural units where  $c = \hbar = 1$ .

## 2 Useful Properties of Commutators

Before doing the actual problems, I will derive the identities which will be useful in solving them as well as in working with commutators in the future. First,

$$[A + B, C] = (A + B)C - C(A + B) = AC - CA + BC - CB = [A, C] + [B, C] \quad (1)$$

and using the fact that  $[A, B] = AB - BA = -(BA - AB) = -[B, A]$ ,  $[C, A + B] = [C, A] + [C, B]$ . Also

$$A[B, C] + [A, C]B = ABC - ACB + ACB - CAB = (AB)C - C(AB) = [AB, C] \quad (2)$$

and similarly  $[C, AB] = A[C, B] + [C, A]B$ .

## 3 Problem 1

a). For an electron in a specific state (i.e. when we know whether it has spin up or spin down) the spin part of the wave function is just

$$\chi = \chi_{\pm 1/2}.$$

Then the expectation values of  $S_z$  and  $S^2$  are

$$\langle S_z \rangle = \chi^* S_z \chi = \pm \frac{1}{2} \chi_{\pm 1/2}^* \chi_{\pm 1/2} = \pm \frac{1}{2}$$

and

$$\langle S^2 \rangle = \chi^* S^2 \chi = \frac{3}{4} \chi_{\pm 1/2}^* \chi_{\pm 1/2} = \frac{3}{4}.$$

Therefore

$$\langle |\cos \theta| \rangle = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

b). This corresponds to  $\theta = 0.96, \pi - 0.96$ .

## 4 Alternate Problem 1

a). I mentioned to some students, that they may interpret the problem as though you are given an electron, which is equally likely to be in either *spin up* or the *spin down* state. In that case

$$\chi = \frac{1}{\sqrt{2}}(\chi_{+1/2} + \chi_{-1/2}).$$

Then

$$\begin{aligned} \langle S_z \rangle &= \chi^* S_z \chi = \frac{1}{2} (\chi_{+1/2}^* \chi_{+1/2} - \chi_{+1/2}^* \chi_{-1/2} + \chi_{-1/2}^* \chi_{+1/2} - \chi_{-1/2}^* \chi_{-1/2}) = \\ &= \frac{1}{2} (\chi_{+1/2}^* \chi_{+1/2} - \chi_{-1/2}^* \chi_{-1/2}) = 0, \end{aligned}$$

where I used the fact that  $\chi_m^* \chi_{m'} = \delta_{mm'}$ . Thus  $\langle |\cos \theta| \rangle = 0$ .

b). This corresponds to  $\theta = \pi/2$ .

## 5 Problem 2

$$A^\dagger = y^\dagger - iq^\dagger = y - iq,$$

where I used the fact that  $y^\dagger = x^\dagger \sqrt{m\omega_0/2} = x \sqrt{m\omega_0/2}$  and similarly for  $q$ . Using Eq. 1 we can compute

$$\begin{aligned} [A, A^\dagger] &= [y + iq, y - iq] = [y, y - iq] + [iq, y - iq] = [y, y] + [y, -iq] + [iq, y] + [iq, -iq] = \\ &= -i[y, q] + i[q, y] + [q, q] = -2i[y, q], \end{aligned}$$

where I used the fact that constants factor out of commutators and anything commutes with itself. Then using the definitions of  $y$  and  $q$  we have  $[y, q] = [x\sqrt{m\omega_0/2}, p/\sqrt{2m\omega_0}] = 1/2[x, p] = i\hbar/2 = i/2$ . Therefore  $[A, A^\dagger] = 1$ . Applying Eqns. 1 and 2 further gives us

$$\begin{aligned} [H, A^\dagger] &= \frac{\omega_0}{2} [A^\dagger A + AA^\dagger, A^\dagger] = \frac{\omega_0}{2} ([A^\dagger A, A^\dagger] + [AA^\dagger, A^\dagger]) = \\ &= \frac{\omega_0}{2} ([A^\dagger, A^\dagger]A + A^\dagger[A, A^\dagger] + A[A, A^\dagger] + [A^\dagger, A^\dagger]A^\dagger) = \\ &= \frac{\omega_0}{2} (A^\dagger[A, A^\dagger] + [A, A^\dagger]A^\dagger) = \omega_0 A^\dagger. \end{aligned}$$

Since I have just proven that  $HA^\dagger - A^\dagger H = \omega_0 A^\dagger$ , we can take the hermitian conjugate of the whole equation and get

$$\omega_0 A = AH^\dagger - H^\dagger A = AH - HA = -[H, A].$$

## 6 Problem 3

First we rewrite  $L_x$  as  $L_x = yp_z - zp_y$  and notice that  $[p_z, z] = -i$  and  $[p_z, x] = [p_z, y] = 0$  (similarly for  $p_y$  and  $p_x$  and all momenta commute with each other). We can then cyclically permute the indicies to get  $L_y = zp_x - xp_z$  (or you can show it the hard way). Then

$$\begin{aligned} [L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] - [zp_y, zp_x] - [yp_z, xp_z] + [zp_y, xp_z] = \\ &= [yp_z, zp_x] + [zp_y, xp_z] = y[p_z, z]p_x + x[z, p_z]p_y = -iyp_x + ixp_y = iL_z. \end{aligned}$$

Similarly

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y] = i[L_y, L_x] - i[L_x, L_y] = -2i[L_x, L_y] = 2L_z,$$

$$[L_-, L_z] = [L_x - iL_y, L_z] = [L_x, L_z] - i[L_y, L_z] = -iL_y + L_x = L_-,$$

where I once again cyclically permuted the indicies in  $[L_x, L_y] = iL_z$  to get the other commutators. Further

$$[L_+, L_z] = [L_x + iL_y, L_z] = [L_x, L_z] + i[L_y, L_z] = -iL_y - L_x = -L_+$$

and

$$[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] = -i(L_x L_y + L_y L_x) + i(L_y L_x + L_x L_y) = 0.$$

Once again cyclically permuting the indices we will similarly get  $[L^2, L_y] = [L^2, L_x] = 0$ . Thus

$$[L^2, L_\pm] = [L^2, L_x] \pm i[L^2, L_y] = 0.$$

## 7 Problem 4

a).

$$L_+^* L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 - i[L_y, L_x] = L^2 - L_z^2 - L_z.$$

Hence

$$(L_+ Y_{ll})^* L_+ Y_{ll} = Y_{ll}^* L_+^* L_+ Y_{ll} = Y_{ll}^* (l(l+1) - l^2 - l) Y_{ll} = 0$$

and therefore  $L_+ Y_{ll} = 0$ .

b).

$$L_-^* L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 + i[L_y, L_x] = L^2 - L_z^2 + L_z.$$

Hence

$$\begin{aligned} 1 &= \int d\Omega Y_{l,m-1}^* Y_{l,m-1} = \frac{1}{|C_-(l,m)|^2} \int d\Omega (L_- Y_{lm})^* L_- Y_{lm} = \frac{1}{|C_-(l,m)|^2} \int d\Omega Y_{lm} L_-^* L_- Y_{lm} = \\ &= \frac{1}{|C_-(l,m)|^2} \int d\Omega Y_{lm}^* (l(l+1) - l^2 + m) Y_{lm} = \frac{l+m}{|C_-(l,m)|^2}. \end{aligned}$$

Thus  $|C_-(l,m)|^2 = l+m$ .

## 8 Problem 5

Once again using the rules for the commutators we have

$$[H, R] = 1/2([LR, R] + [RL, R]) = 1/2([L, R]R + R[L, R]) = E_0 R = HR - RH.$$

Then

$$HRu_E = RHu_E + E_0 Ru_E = (E + E_0)Ru_E.$$

## 9 Problem 6

In this problem (as well as the next one) we will make use of the facts that

$$\int_0^\infty x^n e^{-ax} dx = \frac{n * (n-1) * \dots * 1}{a^{n+1}}$$

and

$$\langle f(r) \rangle = \int R_{nl}(r)^* f(r) R_{nl}(r) r^2 dr \int |Y_{lm}(\theta, \phi)|^2 d\Omega = \int R_{nl}(r)^* f(r) R_{nl}(r) r^2 dr.$$

Thus, using the table on page 245 (and obtaining the fact that  $Z = 1$  from examining the Coulomb potential) we have

$$\begin{aligned} \langle V \rangle &= \langle -\alpha/r \rangle = -\alpha \int R_{21}(r)^* (1/r) R_{21}(r) r^2 dr = \frac{-\alpha}{24} \left(\frac{1}{a_o}\right)^5 \int_0^\infty r^3 e^{-r/a_o} dr = \\ &= \frac{-\alpha}{24} \left(\frac{1}{a_o}\right)^5 6a_o^4 = -\alpha/(4a_o) \end{aligned}$$

and

$$\langle KE \rangle = \langle E \rangle - \langle V \rangle = -\frac{\alpha}{8a_o} - \frac{-\alpha}{4a_o} = \frac{\alpha}{8a_o} = -\langle V \rangle / 2$$

## 10 Problem 7

Using the equations at the beginning of Problem 6 we have

$$\langle r \rangle = \int R_{21}(r)^* r R_{21}(r) r^2 dr = \left(\frac{Z}{a_o}\right)^5 \int_0^\infty r^5 e^{-r/a_o} dr = \left(\frac{Z}{a_o}\right)^5 120 \left(\frac{a_o}{Z}\right)^6 = 120a_o/Z.$$

## 11 Problem 8

We approximate  $r_{av} \simeq 5 * 10^{-16} < a_o$ . Then

$$P_{1,0} = 25 * 10^{-47} 4/a_o^3 = 6.7 * 10^{-15}.$$

b). Also

$$P_{2,1} = 25 * 10^{-47} 25 * 10^{-32} / (24a_o^5) = 6.2 * 10^{-27}.$$

This probability is much smaller because for  $l = 1$  the electron has angular momentum and is circling the nucleus (as opposed to the case of  $l = 0$  when the angular momentum is 0 and the electron is oscillating back and forth through the nucleus). Also for a higher energy state we would expect for the electron to stay farther from the nucleus (on average), but this effect is much less significant (the energies differ only by a factor of 4).